Middle School Students' Steepness and Proportional Reasoning

Diana Cheng
Towson University

Jon R. Star
Harvard University

Suzanne Chapin
Boston University

Abstract

Steepness is a key developmental understanding for slope (Simon & Blume, 1994), and proportional reasoning is closely tied to steepness. Yet the relationship between steepness and proportional reasoning has not been well-studied. This study investigated the relationship between middle school students' proportional reasoning and their solutions to problems involving steepness, which could be measured in a variety of ways including the slopes of inclines. Two tests were administered to students: an adapted version of the Ratio and Proportion Test (Brown et al., 1981) and a Steepness Test. Analysis of data from 413 middle school students showed that 25% of the variability in scores on the Steepness Test could be explained by performance on the Ratio and Proportion Test. The findings of this research contribute to literature on early algebraic reasoning exploring ways that steepness and slope can be made accessible to students.

Keywords: proportional reasoning, slope, steepness

Middle School Students' Reasoning about Steepness

In an era of increasing international scrutiny on the educational preparation of students, the International Association for the Evaluation of Educational Achievement has developed assessments that reflect internationally important concepts that students should learn by grades 4, 8, and 12 (Gonzalez et al., 2008). The content of the eighth grade Trends in International Mathematics and Science Study, implemented every four years at the eighth grade level from 1995 through 2007, especially reveals necessary components of middle grades mathematics in the domains of Number, Algebra, Geometry, and Data and Chance. Of the four domains, Number and Algebra produced the lowest number of students in countries who scored at the "high" benchmark, indicating facility with working with proportional relationships and linear equations (Gonzalez et al., 2008). Thus, there is international interest in improving instruction in proportional reasoning and in algebraic concepts related to linearity.

Slope is a central concept in algebra that is not particularly well understood by students internationally in the middle and secondary grades (Stump, 2001; Yerushalmy, 1997). Slope is fundamentally related to the idea of proportionality which is generally introduced to students in the middle grades. However, this relationship is not explored empirically nor do curricula make the connection explicitly (Lobato & Thanheiser, 2002). Slope is also related to the idea of steepness, a physical characteristic of a line which can be determined visually using an angle or analytically using a proportion (Stump, 1999). The goal of this article is to explore the

Correspondence should be addressed to Diana Cheng, Assistant Professor, Department of Mathematics, Towson University, Towson, Maryland 21252. E-mail: dianasc@gmail.com

relationship between students' understandings of proportional reasoning and steepness, in an effort to shed light upon the learning of slope in the middle grades.

We begin by examining past studies that have focused on students' understandings of proportionality and slope; we then explore the mathematical connections between proportionality and steepness.

Students' Understandings of Proportionality

Proportionality is a multiplicative relationship which can be represented on the coordinate plane as linear functions that pass through the origin (Lobato & Ellis, 2010). Concepts involved in the multiplicative conceptual field include multiplication and division, linear functions and their graphs, rates, ratios, fractions, and rational numbers. These concepts all involve multiplicative relationships between and within quantities. An understanding of the multiplicative conceptual field entails identifying when situations require multiplicative reasoning, particularly distinguishing between the uses of additive and multiplicative reasoning. It also entails being able to perform operations with fractions, such as writing equivalent fractions and solving for an unknown variable in a proportion. With knowledge of concepts in the multiplicative conceptual field, students should be equipped to see connections between modes of representations of multiplicative concepts such as symbols, tables of data, area models, and written descriptions of real-world situations.

A number of parts of the multiplicative conceptual fields are relevant to this study. A proportion is a mathematical relation between quantities that can be represented symbolically as $\frac{a}{b}=\frac{c}{d}$; the ability to mentally process this relation involves proportional reasoning. Reasoning proportionally involves coordinating ratios in multiplicative ways and using rational expressions such as quotients, fractions, and rates.

The most common type of proportional relationship is a ratio. In ratios, there is a multiplicative comparison of two or more quantities. Rates, which are a type of ratio, involve a comparison of two numbers that represent different types of quantities, and there are two measure spaces involved, one for each type of quantity. These types of proportional relationships are called 'associated sets' because each of the quantities is a set and they are associated in a multiplicative way (Lamon, 1993; Marshall, 1993). A common example of a rate is speed, which is expressed using distance over a period of time, such as miles per hour. When graphed on the coordinate plane, all proportional relationships form a straight line that passes through the origin and are often referred to as direct proportional relationships.

Slope refers to a property of a line that represents a proportional relationship. Slope is one of the two ways of measuring a line's steepness; another way to measure the steepness is by using the angle that the line forms with a horizontal line. Using an angular measure of steepness entails looking at one measurement, the number of degrees of the angle. Using the slope to measure steepness entails a multiplicative comparison of two measurements, the lengths of the vertical and horizontal differences. A diagram of the connections between these ideas is shown in Figure 1.

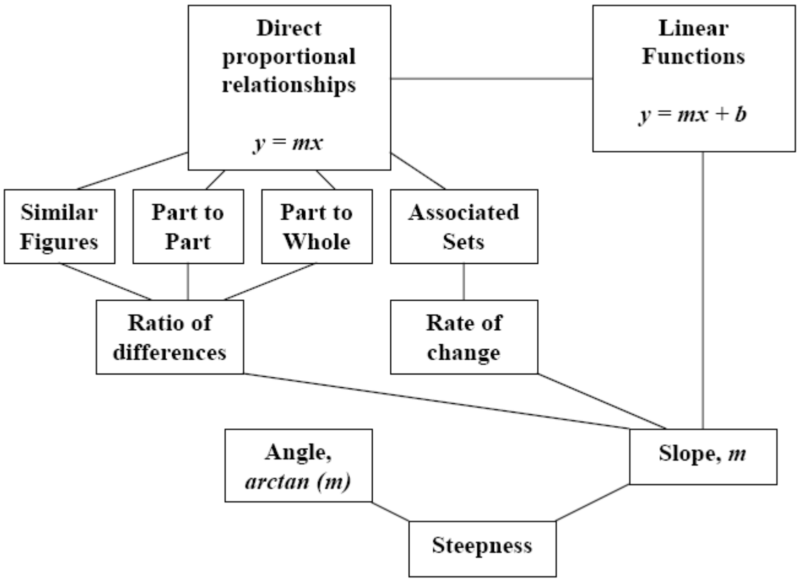


Figure 1. Concepts in the multiplicative conceptual field related to proportional reasoning and slope.

Students' successes on problems involving proportionality are highly dependent upon the contexts in which the problems are situated (Harel, Behr, Post, & Lesh, 1991; Tourniaire & Pulos, 1985). For example, students' unfamiliarity with contexts such as photocopying and enlargements of two-dimensional figures may prevent them from using proportional reasoning to find corresponding dimensions when it is appropriate (DeBock, Verschaffel, & Janssens, 1998; Tourniaire & Pulos, 1985). Also, the presence of a liquid mixture context tends to confuse students, whereas situations in which discrete values are used are easier for students to understand (Harel, Behr, Lesh, & Post, 1994; Tourniaire & Pulos, 1985).

Student success on problems involving proportionality is also dependent upon the structural difficulty levels of the values present in the problems (Harel, Behr, Post, & Lesh, 1994; Moss & Case, 1999; Noelting, 1980a). Students tend to have greater success on comparison proportion-related problems where two of the corresponding values between the ratios involved are the same (e.g., 3/4 and 3/5 where the numerators of the fractions are the same, or 1/3 and 2/3, where the denominators of the fractions are the same), when the ratios have the same value (e.g., $\frac{2}{2}$ and $\frac{3}{3}$), or when one ratio is equivalent to $\frac{1}{1}$ and the other is not (e.g., $\frac{2}{2}$ and $\frac{3}{4}$). Students tend to have trouble solving problems where there is a difference of one between corresponding values in the ratios (e.g., 5/7 and 6/9 where the numerators differ by one) or when none of the values in the ratio are multiples of each other (e.g., $\frac{5}{2}$ vs. $\frac{7}{3}$ where 5, 2, 7 and 3 are all relatively prime).

Students' common errors when solving proportional reasoning problems include using incorrect or irrelevant data in their computations, and using additive reasoning instead of multiplicative reasoning (Hart, 1981; Lobato & Thanheiser, 2002). In addition, when students rigidly apply rules that they do not understand, they often are unsuccessful at solving proportion problems (Streefland, 1991).

Researchers have explored various interventions to try to help students avoid these errors and develop understanding of proportionality. For example, the Rational Number Project developed instructional materials that help students pay attention to both the numerator and denominator of each fraction as opposed to only one or the other, and use a reference point to compare fractions to help students avoid using additive strategies (e.g., Behr, Wachsmuth, Post, & Lesh, 1984). However, in many studies, researchers have found that students' difficulties with proportional reasoning are deeply held and resistant to change (Adjiage & Pluvinage, 2007; Behr et al., 1984).

Students' Understandings of Slope

Proportionality and slope are related because slope is a ratio. The slope of a line is the amount of a line's vertical change for each horizontal unit that it covers. An understanding of slope is fundamental to understanding properties of the simplest algebraic functions, lines.

Students exhibit many difficulties with understanding slope. For example, students frequently have trouble determining which data are relevant when considering slope. High school students who are given the "rise over run" formula for slope may have difficulty determining which values to assign to the rise and the run if they do not understand the formula's meaning (Lobato, 1996). High school students sometimes confuse a line's y-intercept with the line's slope, when these two characteristics of a line have very different meanings (Moschkovich, 1996). Students also commonly believe that only one measurement, rather than two, needs to be taken in order to compute slope (Moyer, Cai, & Grampp, 1997). When mathematics learners successfully identify the two sets of measurements that need to be considered when computing the slope of a physical object, they may not always relate the measurements using a ratio. A common error exhibited by preservice elementary teachers is to subtract one measurement from the other (Simon & Blume, 1994), thus working with the ratio additively rather multiplicatively.

Various interventions have been designed to improve students' understanding of slope. For example, Moschkovich (1998) reported a study where high school students used graphing software to learn about the graphs of linear functions written in the form, $y = mx + b$. One of the student conceptions that was observed was that m denoted the x-intercept; this conception showed up less frequently in the posttest than on the pretest, however it did appear on both tests. Lobato, Ellis and Munos (2003) found that at the end of a five week unit on slope using a curriculum designed by the Core Plus Math Project (Schoen & Hirsch, 2003) which asked students to calculate a quotient to find the slope, many high school students still viewed m as a difference rather than a ratio. In an attempt to help students understand slope when learning algebra, Simon and Blume (1994) suggested that students as well as perservice teachers understand how to find a measure of the steepness of a physical incline prior to encountering the functional concept of slope. Lobato and Siebert (2002) empirically confirmed that instruction can help a student in a beginning algebra class progress from viewing steepness of a wheelchair ramp solely as an angle, to seeing how both height and length of the ramp affect steepness, to coordinating height and length in a ratio, the slope.

Connections between Proportionality, Steepness and Slope

A premise of this paper is that one important way that students' difficulties with both proportionality and slope might be addressed is by focusing on better connections between these two pivotal concepts while examining the physical property of steepness. Teuscher and Reys

(2010) suggest that teachers and curriculum developers deliberately help middle school students distinguish between slope and steepness, in preparation for learning about rate of change of functions in precalculus in high school. One recommendation was to use rooftops as a physical introduction to slope and steepness – and to focus on the idea that while the two sides of a roof will have the same steepness, one side of the roof's slope will be positive and the other side of the roof's slope will be negative. Despite this suggestion, Stanton and Moore-Russo (2012) found that although all of the United States state curriculum standards required that the concept of slope be addressed, very seldom (only 6 states out of 50) did state standards suggest examining physical properties of steepness to help students learn about slope.

Mathematically, there are connections between proportional reasoning, steepness and slope. For instance, in tasks involving proportional reasoning, fractional values often need to be compared. On lines drawn on the same coordinate axes, lines with the same steepness will have slopes that are equivalent fractions, and lines that are steeper have larger slopes. Hattikudur, Prather, Asquith, Knuth, and Nathan (2012) examined middle school students' solutions to rate-related proportion problems which also asked students to graph their responses. They found that some students who were able to correctly identify the rate or slope subsequently made arithmetic errors in calculating points on a linear graph (for instance, students graphed the points (2,2), (3,4), and (4,7) to correspond to a line with slope 2). This naturally leads to questions regarding whether these students truly understood the proportions that informed their point-to-point slope calculations, or whether students understood what a constant slope of a line meant with respect to the steepness of the line segments connecting the points.

Even though there are mathematical connections between proportional reasoning, slope, and steepness, additional research can be done to explore whether these connections exist for students, and whether augmenting students' knowledge about proportional reasoning is linked to increased knowledge of steepness and slope. The present study explores the extent to which proportional reasoning and steepness are related for students. If such a relationship exists between students' conceptions of proportionality and steepness, there may be implications for the development of curricula that introduce slope to students using the idea of steepness.

Research Questions

The purpose of this study is to determine what relationships exist between students' solutions to steepness problems and students' solutions to proportion problems. More specifically, the study was designed to answer the following questions:

1. To what extent are students able to successfully solve steepness problems? To what extent does their success vary based on the context in which the problem is situated and the structural difficulty level of the problem?
2. What is the relationship between students' proportional reasoning scores and their steepness scores?

Method

As will be described in greater depth below, a large scale survey consisting of two tests (a Ratio and Proportion Test and a Steepness Test) was administered to 413 middle school students.

Participants

The sample for this study consisted of 413 students in grades 6, 7 and 8 who attended one public middle school. The dominant ethnicities of the students at the school were white (62.3%

of the students) and Asian (25.8% of the students). The participants in the study were enrolled in courses titled Grade 6 Math, Extended Mathematics, Extended Algebra, Prealgebra, and Algebra 1. The numbers of participants by grade and gender are provided in Table 1. All the students of seven out of the eleven mathematics teachers in the school participated in the study.

Table 1
Survey Study Sample by Grade and Gender

Grade	Male	Female	Unlisted	Total
6	77	72	3	152
7	61	53	1	115
8	70	76	0	146
Total	208	201	4	413

Procedure

Teachers handed participants both instruments together, with the Ratio and Proportion Test and the Steepness Test stapled together consecutively in one package. All students finished within the allotted time of 60 minutes. Participants did not receive incentives for participating in the study and were told that their participation would not impact their mathematics course grades. The first author had a prior relationship with the school and the mathematics teachers; teachers mentioned to participants that this was part of a research study and they expected students to try their best.

Instruments

To assess the middle school students' levels of proportional reasoning, the Ratio and Proportion Test from the *Increasing Competence and Confidence in Algebra and Multiplicative Structures Test R* (Brown et. al., 1981), was used. There were eight problem settings, and a total of 20 problems on the Ratio and Proportion Test. Each problem on the test was rated at one of four structural difficulty levels based upon the numbers involved. Participants attained levels by solving correctly 60% of the problems in that level. The highest level assigned to a participant was the one for which he/she attained all prior levels. Each problem on the test was scored for correctness; correct solutions received one point and incorrect solutions received zero points. Correctness on this open-ended test indicates that participants had a productive way of solving the proportion-related problem. The participants in this study could be expected to correctly answer all of the items on the test. In addition, they had been introduced to nonstandard units of measurement in elementary school.

To assess the middle school students' responses to steepness problems, the Steepness Test was developed, drawing on past research and piloting by the first author (Cheng & Sabinin, 2008, 2009) as well as prior research by Noelting (1980a, 1980b). Given the recommendation of Moyer et al. (1997) that instruction on slope should begin with comparison activities, comparison problems were used on the Steepness Test. The test included 24 problems that asked participants to determine which of two drawings was steeper. Each problem asked participants to compare the steepness of two inclines and had three answer choices: 1) the left incline is steeper, 2) the right incline is steeper, or 3) the inclines have the same steepness. Correct responses earned 1 point and incorrect responses earned 0 points. Students' correct responses indicated that they found productive ways of solving the steepness problems, although the strategies may have been only applicable to specific contexts or structural difficulties.

There were three types of problem contexts: two situated the problem of steepness as an incline and one presented it as a mathematical problem. Within each problem context there were eight problems, all grouped together. The two inclines were roofs and staircases. All drawings of roofs and staircases were shown on grid paper. The mathematical problems on the Steepness Test showed two lines in Quadrant 1, and each of them started at the origin. To solve these problems, participants needed to compare the steepness of the roofs, staircases, and lines. All diagrams were presented on coordinate grids with homogeneous axes.

The development of the Steepness Test was based on work by Noelting (1980a), who empirically ordered comparison proportion-related problems in terms of difficulty based upon the numbers given in the problem contexts. The slope pairings on the Steepness Test were taken directly from those used in Noelting's survey that asked which of two orange juice mixes was more 'orangey': the one with *a* cups of orange juice and *b* cups of water, or the one with *c* cups of orange juice and *d* cups of water, where *a, b, c,* and *d* represent integers that were given to the participants (Noelting, 1980b). For instance, a mix with 1 cup of orange juice and 2 cups of water would be more orangey than a second mix with 1 cup of orange juice and 4 cups of water because the second mix is more diluted with water. However a third mix with 2 cups of orange juice and 4 cups of water would have the same orangey-ness as the first mix because the ratios between orange juice and water are equivalent.

The slopes presented on the Steepness Test had values of $\frac{a}{b}$ and $\frac{c}{d}$ where the numerators were numbers of cups of orange juice from Noelting's (1980a) problems, and the denominators were numbers of cups of water. For instance, one of Noelting's problems asked which mix tasted more orangey, the orange juice mix with 2 cups of orange juice and 1 cup of water or the orange juice mix with 1 cup of orange juice and 2 cups of water. The corresponding slopes used on the Steepness Test were $\frac{2}{1}$ and $\frac{1}{2}$, and Problem 1 on the Steepness Test showed a pair of roofs with these slopes (see Appendix A).

Each set of slopes was used in all three contexts on the Steepness Test. Each pair of slopes was selected from one of the eight stages of proportional reasoning which Noelting (1980b) delineated. In order to mirror the language used for the Ratio and Proportion test, we labeled Noelting's stages as structural difficulty levels, or difficulty levels, because the stages were also created based upon the numerical structures of the problems.

Participants who correctly answered 2 of the 3 problems in a particular level were said to have attained that level, to closely match the 60% criteria set on the Ratio and Proportion Test. The level that participants were assigned was the highest level for which they attained all prior levels. The pairings of slopes were ordered in levels of difficulty as determined by Noelting (1980b), with problems in the lowest structural difficulty level (level 1) using slopes of $\frac{2}{1}$ and $\frac{1}{2}$, whose arctangents have an angular difference of 36 degrees, and problems in the highest structural difficulty level (level 8) using slopes of $\frac{2}{5}$ and $\frac{3}{7}$, whose arctangents have an angular difference of 1.4 degrees.

Two of the structural difficulty levels (levels 4 & 5) depicted inclines of the same steepness. For the remaining six difficulty levels, the Steepness Test showed pairs of inclines whose steeper incline relative to the bottom of the page randomly alternated between being depicted on the left or on the right. Participants likely used a variety of strategies to solve the problems; for those who compared the angles between the two inclines, this strategy would likely yield an inconsistently correct answer because the angular difference between inclines in the higher structural difficulty levels was difficult for the naked eye to judge.

Table 2 displays the problem difficulty levels, the slopes for each level, and the differences in angle measure between the two inclines with the given slopes. Since the difficulty levels were empirically developed from students' proportional reasoning, it is theoretically possible that students could attain a subsequent difficulty level without having attained the prior difficulty level.

Table 2
Level, Slope, and Angular Difference of Steepness Test Items

Difficulty Level	Slopes of Two Inclines	Angular Difference Between Inclines
1	½ vs. 2/1	36.8
2	1/1 vs. 2/1	18.4
3	2/2 vs. 4/3	8.1
4	2/2 vs. 3/3	0
5	2/4 vs. 1/2	0
6	3/1 vs. 5/2	3.4
7	3/2 vs. 4/3	3.2
8	2/5 vs. 3/7	1.4

Within each set of eight problems (roofs, staircases, lines), the order of the problems based on structural difficulty level was randomly determined. The order in which the groups of problems based on context (roofs, staircases, lines) were arranged on the Steepness Test was not chosen at random, and may have had an effect on students' responses to the problems. Since the roof and the line problems present continuous data whereas the staircase problems involve discrete data, the stairs were placed in the middle of the instrument. Hence, the eight roof problems were presented first, followed by eight staircase problems, followed by eight line problems. The complete assessment is provided in Appendix A.

Since the steepness problems were comparison problems modeled after Noelting's (1980a, 1980b) problems, it was possible to correctly answer the problems by random guessing or by using irrelevant information. Thus, the numbers of problems which could be expected to be answered correctly solely by random guessing are reported alongside data of participants' scores.

Participants in the study encountered roofs and stairs in their daily lives, as a majority of houses in the community had a roof on top and stairs inside. Participants were likely to have encountered escalators and stairs at local malls and buildings. Since the survey was conducted in a northeastern state of the United States, many of the participants were likely to have had experience sledding down hills or skiing down slopes during the winter. Although participants may have had experience with physical inclines, they may not have considered comparing them in quantitative ways or considered stairs from their side views.

Instrument Validity and Reliability Analysis

Content validity of the Ratio and Proportion Test was established during the CSMS project (Hart, 1981) which constructed the test items in consultation with British textbooks, mathematical experts, and mathematics education experts. Internal reliability of the Ratio and Proportion Test was established using Kruskal's Gamma Test which found the gamma coefficients between student scores on the Ratio and Proportion Test and the following tests: Algebra 0.763, Graphs 0.790, Fractions 1.839, Measurement 0.790, Decimals 0.800, Positive and Negative Numbers 0.596, and Reflections / Rotations 0.591 (Brown et al., 1981). These data show that there was high agreement on test scores on the topics that were closely associated with ratios, and lower agreement on test scores on topics that were less related to ratios.

Five mathematicians and mathematics educators established content validity of the Steepness Test; they each agreed that the Steepness Test items measured knowledge of steepness. In order to establish the reliability of the Steepness Test, 134 participants in grades 6, 7, and 8 took the test twice on two consecutive class days. A paired samples correlation showed that the correlation between the first and second administration total scores was 0.790 ($p < 0.001$). A paired samples t-test showed no evidence of significant difference between total scores on the two administrations ($df = 133$, $p = 0.720$).

All participants' responses to each test item on both tests were entered into a spreadsheet that was programmed to score the tests. Several mathematics educators verified that the formulas used in the spreadsheet were correct.

Results

This study examined the extent to which students were able to successfully solve steepness problems based upon the problem's context and structural difficulty, and examined the relationship between students' proportional reasoning and steepness scores. Data will first be presented on participants' Steepness Test scores by context and structural difficulty level, followed by a presentation of results on participants' Ratio and Proportion Test results. Finally, participants' achievement on the Ratio and Proportion Test will be compared to their achievement on the Steepness Test.

Steepness Test Results

Data from the administration of the Steepness Test were analyzed to examine participants' solutions to problems involving steepness. The participants' mean score was 16.03 (*SD = 3.169*) out of 24 questions. Since each of the comparison problems on the Steepness Test had a choice of three responses (left is steeper, right is steeper, or both are the same steepness), on average participants could have answered 8 out of 24 problems correctly by randomly guessing. The 16.03 mean score attained by participants indicates that, on average, they performed twice as well as they would have if they had randomly guessed.

In order to investigate whether solving steepness problems is related to context, means for each of the contexts were computed. On average, participants correctly answered 4.48 (*SD = 1.442*) of the 8 staircase problems, 5.26 (*SD = 1.254*) of the 8 roof problems, and 6.29 (*SD = 1.460*) of the line problems. Scores attained by random guessing in the multiple choice setting would average 2.67 for each context. Results of paired *t*-tests show that there was evidence of a significant difference between performances on each pair of contexts ($p < 0.001$ for each pair).

A brief summary of the scoring data on participants is presented in Table 3 by context and structural difficulty level.

Table 3
Survey Participants' Performance on the Steepness Test by Context and Level (n = 413)

Structural Difficulty Levels	Contexts		
	Staircases	Roofs	Lines
Level 1	97%	97%	96%
Level 2	83%	97%	95%
Level 3	79%	83%	80%
Level 4	45%	81%	76%
Level 5	44%	76%	66%
Level 6	35%	33%	96%
Level 7	24%	30%	69%
Level 8	40%	30%	51%

In order to investigate whether success in solving steepness problems was related to problem difficulty, participants' performance on problems in the eight structural difficulty levels was examined. Participants attained a level by correctly answering two out of three problems at that level, a criteria adapted from structural difficulty level attainment on the Ratio and Proportion Test. The frequency at which participants attained each of the eight levels is shown in Table 4.

Table 4
Percentages of Survey Participants Who Attained Each Steepness Test Structural Difficulty Level

Level	Percentage of Participants who Attained Level
1	97%
2	92%
3	81%
4	67%
5	62%
6	55%
7	41%
8	40%

Participants' mean scores were also determined by grade, and the results are presented in Table 5. The scores presented are out of 3 possible correct responses for each structural difficulty level. On average, grade 6 participants answered 2.53 problems correctly in level 1, whereas grade 8 participants answered 2.93 problems correctly in that level.

Table 5
Survey Participants' Mean Scores and Standard Deviations by Structural Difficulty and Grade

Steepness Difficulty Level	Grade 6 Mean	Grade 6 Std. Dev.	Grade 7 Mean	Grade 7 Std. Dev.	Grade 8 Mean	Grade 8 Std. Dev.
1	2.53	0.72	2.91	0.28	2.93	0.35
2	2.66	0.61	2.79	0.45	2.78	0.45
3	2.26	0.77	2.44	0.72	2.41	0.73
4	2.01	0.89	1.94	1.01	2.21	0.73
5	1.75	0.97	1.8	1.07	2.1	0.88
6	1.51	0.81	1.65	0.78	1.71	0.78
7	1.32	0.87	1.26	0.85	1.27	0.96
8	1.55	0.99	1.17	0.87	1.13	0.98

Further analyses of participants' performance by grade on the steepness problems were conducted using Tukey's Honestly Significant Difference Test in conjunction with a repeated measures ANOVA, using structural difficulty level as a within-subject factor and grade as a between-subject factor. The analyses showed a significant difference in performance on all difficulty levels for grades 6 and 8 (p = 0.028). There were no significant differences between performances of participants in grades 6 and 7 (p = 0.604) on the difficulty levels, nor between performances of participants in grades 7 and 8 (p = 0.317). In summary, participants' performances on the eight structural difficulty levels generally decreased as the structural difficulty levels increased, with only a few exceptions. Participants in grades 6 and 8 had significantly different performances on the structural difficulty levels.

In studies that investigate the cognitive demands for solving tasks of various structural difficulties, researchers have found that tasks requiring consideration of fewer comparisons are easier for participants to solve. In particular, taking into account four values simultaneously, which is required for proportional reasoning, is cognitively more complex than only taking into account one quantity. For instance, on a balance scale task used in research beginning with Inhelder and Piaget (1958) and subsequently used by other researchers (Halford, Andrews, Dalton, Boag, & Zielinski, 2002; Siegler, 1976), it was found that problems were more likely to be successfully solved if one of the two variables, weight or distance from fulcrum, was held constant. Siegler (1976) hypothesized that this is the case because participants only need to take into account one quantity rather than two. When both the weight and distance varied, some participants in Siegler's study still took into consideration only one of these values.

The findings of the survey portion of the present study are consistent with the findings of research on relational complexity. Steepness Test Level 2 problems involved a comparison of one dimension; inclines with slopes of $\frac{1}{1}$ and $\frac{2}{1}$ were compared. Only one variable differed in this case and approximately 92% of the Level 2 problems in the survey study were solved correctly.

The findings of the present study confirm that the progression which Noelting (1980a) found in his proportional reasoning context is also relevant for visually represented steepness comparison problems in the contexts of staircases, roofs, and lines. Percentages of participants who attained each of the structural difficulty levels on the Steepness Test decreased as the structural difficulty levels increased, indicating that the problems in the higher structural difficulty levels were more difficult.

In order to investigate whether success on solving steepness problems was dependent on the interaction of context and difficulty level, a per-problem analysis was conducted to examine the association between context, structural difficulty level, and the interaction between context and structural difficulty level with the probability of a correct response on the Steepness Test. A per-problem analysis was used because each context and structural difficulty level combination was represented by only one problem on the Steepness Test. The outcome for these analyses was whether or not a problem was correctly answered, and differences in the probability of a correct answer were described through odds ratios that were obtained from multiple logistic regression.

The odds ratios presented in Table 6 compare the odds of a correct response for staircase or line problems to that of roof problems in a particular difficulty level. For instance, a participant who correctly answered the level 4 roofs problem would correctly answer the level 4 staircase problem with probability 20% and the level 4 lines problem with probability 74%.

Table 6

Associations Between Context, Structural Difficulty Level and Probability of a Correct Response

		Staircases		Lines	
		Odds Ratio	p value	Odds Ratio	p value
Steepness Test Structural Difficulty Levels	Level 1				
	Level 2	0.13	< 0.001		
	Level 3				
	Level 4	0.20	0.001	0.74	0.035
	Level 5	0.26	< 0.001	0.62	< 0.001
	Level 6			50	< 0.001
	Level 7	0.74	0.026	5.25	< 0.001
	Level 8	1.56	0.001	2.47	< 0.001

Note. Only significant odds ratios are reported

An investigation was conducted to determine whether participants in the three grades had different performances with respect to context on the Steepness Test problems. Participants in all three grades scored the lowest on staircase problems, next lowest on roof problems, and highest on line problems. On average, participants in Grade 6 correctly answered 4.2 out of the eight staircase problems, whereas participants in Grade 8 on average, correctly answered 4.66 of these problems. Also, on average, participants in Grade 6 correctly answered 6.14 line problems compared to 6.55 problems correctly answered by Grade 8 participants. The mean subscores for each of grades 6, 7, and 8 are presented in Table 7.

Table 7
Participants' Steepness Test Means and Standard Deviations by Grade and Context

	Grade 6 (n=152)		Grade 7 (n=115)		Grade 8 (n=146)	
Context	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
Staircases	4.2	1.44	4.62	1.36	4.66	1.48
Roofs	5.26	1.28	5.19	1.19	5.32	1.28
Lines	6.14	1.54	6.16	1.53	6.55	1.28

Further analyses of participants' performance by grade with respect to the three contexts were conducted using Tukey's Honestly Significant Difference Test in conjunction with a repeated measures ANOVA, using context as a within-subject factor and grade as a between-subject factor. The analyses showed a significant difference in performance on all contexts for grades 6 and 8 ($p = 0.028$). There were no significant differences in performance on contexts between grades 6 and 7 ($p = 0.604$) nor between grades 7 and 8 ($p = 0.604$). In summary, the investigation of participants' performances on the three contexts revealed that participants in all three grades performed the best on the line problems, followed by roof problems and staircase problems respectively. Participants in grade 8 performed significantly better than participants in grade 6 on all three contexts.

In summary, significant differences were found in participants' performances on the three contexts as well as in participants' performances on the eight structural difficulty levels. However, a limitation of the study is that the Steepness Test consisted of only 24 items, with only one problem representing each different pairing of context and structural difficulty level. In order to draw conclusions about the interaction of context and structural difficulty level and its impact on participants' successes in solving problems involving steepness, multiple problems with the same pairing of context and structural difficulty level should be used.

Ratio and Proportion Test Results

On the Ratio and Proportion Test, each of the 20 problems was associated with a structural difficulty level ranging from 1 to 4. Participants attained a level by correctly answering at least 60% of the problems in a structural difficulty level, meeting this requirement for all of the lower structural difficulty levels, and failing to meet this requirement for the next higher structural difficulty level. Approximately 6% of participants failed to attain Proportional Reasoning Level 1, and were assigned Level 0 on the Ratio and Proportion Test. The percentages of participants who attained the five levels ranging from 0 to 4 on the Ratio and Proportion Test are presented in Table 8.

Table 8

Percentages of Survey Participants' Levels on the Ratio and Proportion Test

Proportional Reasoning Level	Percentage of Participants (N=413) Who Attained Level
PR 0	6%
PR 1	27%
PR 2	25%
PR 3	34%
PR 4	9%

Ratio and Proportion Test and Steepness Test Results

To investigate the relationship between the participants' proportional reasoning levels and their success on the Steepness Test problems by context, several means were computed. There were eight Steepness Test problems in each of the three contexts, Staircases, Roofs, and Lines. There was a gradual increase in the means for each of the three Steepness Test contexts as proportional reasoning levels increased. As shown in Table 9, participants in all proportional reasoning levels correctly answered the fewest staircase problems and the most line problems.

Table 9

Participants' Mean Scores by Proportional Reasoning Level and Steepness Context

Proportional Reasoning Level	Steepness Test Context					
	Staircases		Roofs		Lines	
	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
PR 0	3.78	1.09	4.78	1.04	5.22	1.54
PR 1	3.92	1.19	4.81	1.01	5.68	1.53
PR 2	4.42	1.31	5.03	1.16	6.32	1.32
PR 3	4.83	1.46	5.65	1.26	6.78	1.21
PR 4	5.42	1.72	6.05	1.43	6.82	1.49

To investigate the relationship between participants' proportional reasoning levels and their success on Steepness Test problems by structural difficulty level, percentages of participants in each of the five proportional reasoning levels who attained each of eight Steepness Test Levels were computed. Table 10 below shows the percentages of participants who attained each grouping of levels on the Ratio and Proportion Test and the Steepness Test.

Table 10
Percentages of Participants by Ratio and Proportion Level and Steepness Test Level

Proportional Reasoning Level	Steepness Test Level S 1	S 2	S 3	S 4	S 5	S 6	S 7	S 8
PR 0	96%	91%	83%	65%	44%	26%	17%	22%
PR 1	96%	96%	75%	74%	60%	41%	15%	36%
PR 2	96%	98%	89%	68%	65%	49%	34%	46%
PR 3	94%	98%	94%	84%	78%	57%	51%	48%
PR 4	97%	97%	92%	92%	84%	66%	61%	45%

Participants who attained the higher proportional reasoning levels were generally able to attain more of the Steepness Test levels. For instance, approximately 75% or more of the participants who attained Proportional Reasoning levels 0, 1, and 2 attained Steepness Test levels 1, 2, and 3. Using the same benchmark of 75%, approximately 75% or more of the participants who attained Proportional Reasoning levels 3 and 4 attained Steepness Test levels 1, 2, 3, 4, and 5.

To determine the correlation between the 413 participants' total scores on the Ratio and Proportion Test and the Steepness Test, linear regression was performed. A table with each participant's scores is shown in Figure 2.

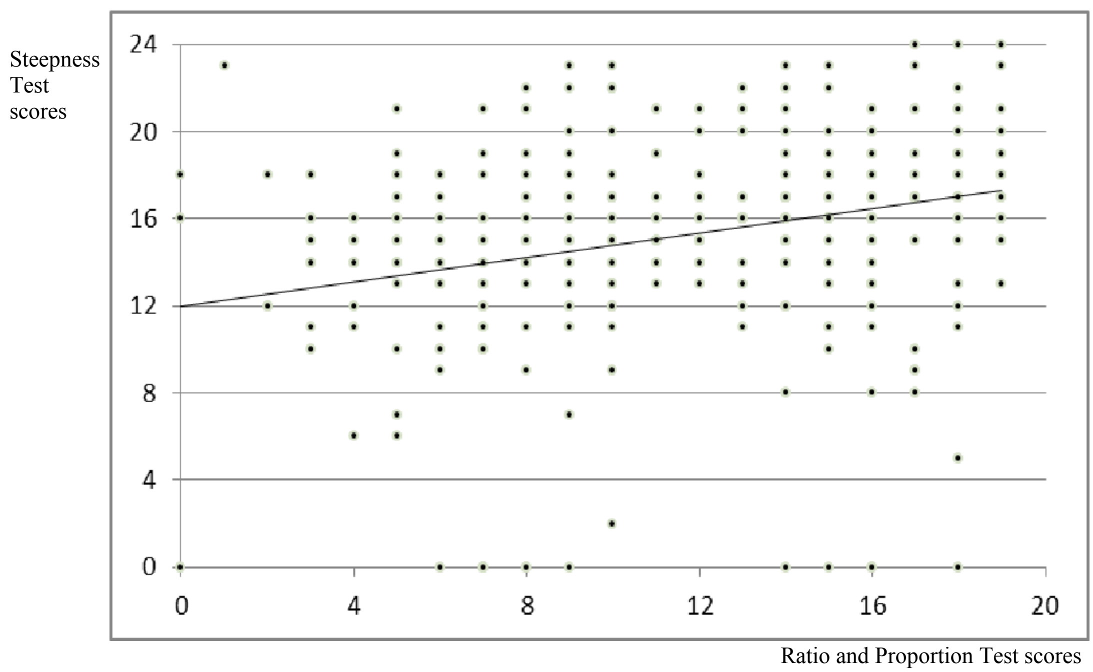


Figure 2. Steepness Test score as predicted by Ratio and Proportion Test score

The results of the regression analysis indicate that there was a positive correlation between participants' Ratio and Proportion Test total scores and participants' Steepness Test total scores. The equation predicting Steepness Test score *(S)* from Ratio and Proportion Test score *(R)* was $S = 0.345\ R + 12.003$ ($p < 0.001$ for the slope and *y*-intercept). This indicates that on average, participants scored a base score of 12.003 on the Steepness Test and for each problem answered correctly on the Ratio and Proportion Test, they answered 0.345 Steepness Test problems correctly. From the linear regression, the R-squared value is 0.254, indicating that 25.4% of the variability in Steepness Test correctness can be explained by Ratio and Proportion Test correctness. This is a large percentage, considering the numerous other factors which may have contributed to students' scores on the Steepness Test, including their consideration of height instead of steepness, attitudes and beliefs towards mathematics and their mathematics teachers, their prior exposure to slope, their overall achievement in mathematics, their use of mathematics at home, their test-taking abilities, their motivation to take the tests, and so on.

Linear regressions were also performed on subsets of the data by grade. The equations predicting Steepness Test score from the Ratio and Proportion Test score were: $S = 0.382\ R + 11.290$ for grade 6 (R-squared value is 0.274), $S = 0.316\ R + 12.296$ for grade 7 (R-squared value is 0.251), and $S = 0.321\ R + 12.642$ for grade 8 (R-squared value is 0.232). All of the slopes and y-intercepts found from the linear regressions by grade were significant ($p < 0.001$).

In summary, there was a relationship between participants' scores on the Ratio and Proportion Test and the Steepness Test. Participants who attained higher proportional reasoning levels generally scored higher on Steepness Test problems when they were grouped by context. Participants who attained higher proportional reasoning levels had higher frequencies of attaining Steepness Test Levels.

Discussion

There are several possible explanations for participants' differing performances on the Steepness Test contexts and structural difficulty levels. Several research studies have shown that the ability to reason proportionally is highly dependent upon context, that some tasks facilitate students' reasoning proportionally more than others, and that students' familiarity with contexts tends to help them solve proportional reasoning problems (Bright, Joyner & Wallis, 2003; Tourniaire & Pulos, 1985). In this study, participants in grades 6, 7, and 8 answered steepness problems about lines correctly most frequently, followed by roofs and staircases. On roof and staircase problems, participants may have been unclear as to what physical features to look for in determining relative steepness. Mitchelmore and White (2000) hypothesized that the sloping edges of a hill depicted in their diagrams helped their grades 2-8 research participants identify similarities between the hill and a standard angle. A similar effect may have taken place in the present study. The staircase problems did not explicitly contain lines whose steepness could be compared, whereas the roof and line problems did contain lines whose steepness could be directly compared. Additionally, roof problems contained more lines than necessary (e.g., the rectangular houses underneath the roofs) whereas the line problems only depicted relevant lines.

This raises additional questions about the kind of familiarity that students need to have about a context in order to reason about steepness. Students generally encounter staircases earlier in their lives than they encounter lines drawn on a coordinate plane. Why, then, would students have the most difficulty solving steepness problems involving staircases? Even though students often climb physical sets of staircases in everyday life, it may be difficult for students to judge the steepness of staircases due to irrelevant data which they may consider as factors contributing to steepness. Another possible explanation for limited success on staircase problems could be that some participants may have more difficulty with solving visual problems depicted with discrete units. Boyer, Levine and Huttenlocher (2008) found that students in kindergarten through fourth grade had more difficulty solving proportional reasoning problems represented visually when the pictures had discrete units demarcated on them than when the pictures did not have units marked. The results of this study are consistent with Boyer, Levine and Huttenlocher's findings that proportional reasoning problems whose visual depictions are more continuous in nature are easier for participants to correctly solve.

In addition to an investigation of participants' performances with respect to context, an investigation of participants' performance with respect to structural difficulty levels was conducted. The percentages of participants who attained each of structural difficulty levels 1 through 8 on the Steepness Test decreased as the structural difficulty levels increased, indicating that the problems in the higher structural difficulty levels were more difficult. The slopes for each of the structural difficulty levels were chosen based upon Noelting's (1980a) empirically derived progression of difficulty that participants experienced when solving comparison problems involving orange juice concentrations. The percentages of Noelting's participants aged 6 though 16 who correctly solved the comparison problems in each of the structural difficulty levels are listed below in Table 11, alongside the percentages of participants in the present study who attained each of the corresponding Steepness Test difficulty levels.

Table 11
Percentages of Participants by Study and Structural Difficulty Level

Steepness Test Difficulty Levels	Percentages of Noelting's Participants	Percentages of Present Study Participants
Level 1	99%	97%
Level 2	95%	92%
Level 3	93%	81%
Level 4	78%	67%
Level 5	68%	62%
Level 6	57%	55%
Level 7	28%	41%
Level 8	22%	40%

The findings of the present study confirm that the progression which Noelting found in his experimental context is also relevant for visually represented comparison problems in the contexts of staircases, roofs, and lines.

In studies that investigate the cognitive demands for solving tasks of various structural difficulties, researchers have found that tasks requiring consideration of fewer quantities were easier for participants to solve. In particular, taking into account four quantities simultaneously, which is required for proportional reasoning, is cognitively more complex that only taking into account one quantity. For instance, on a balance scale task used in research beginning with Inhelder and Piaget (1958) and subsequently used by other researchers (e.g., Siegler, 1976), it was found that problems were more likely to be successfully solved if one of the two variables, weight or distance from fulcrum, was held constant. Siegler (1976) hypothesized that this is the case because participants only need to take into account one quantity rather than two. When both the weight and distance varied, some participants in Siegler's study still took into consideration only one of these quantities. The findings of the present survey are consistent with the findings of research on relational complexity. Steepness Test level 2 problems involved a comparison of one dimension; objects with slopes of 1/1 and 2/1 were compared. Only one variable differed in this case and approximately 92% of the Level 2 problems in the survey study were solved correctly. Level 8 Steepness Test problems, which compared objects with slopes of 3/7 and 2/5, were the most difficult for the survey participants.

This research shows that there is a relationship between the correctness of the participants' proportional reasoning and their steepness responses. Unlike many of the accounts of student learning in this area that have used qualitative methods (e.g., Yerushalmy, 1997), this study presents quantitative research in a large-scale setting. This work builds on research by Lobato and Thanheiser (2002) and Stump (2001), who suggest that an understanding of quantifying steepness using a ratio as the measure of a physical incline may help students develop a deeper understanding of proportions, steepness, and slope than would be developed simply by telling them to use a formula to find the slope of a line. The outcomes of this study suggest the development of a pre-algebraic curriculum which introduces proportional reasoning, steepness, and slope as related concepts and focuses on students' abilities to translate between multiple representations of these concepts.

One such set of activities was created by Cheng (2010), who suggests that in the middle grades, the comparison of fractions should be taught in conjunction with the graphical representation of lines with corresponding slopes on a coordinate plane. By comparing the steepness of lines with respect to each other, and by receiving appropriate guidance from their mathematics teachers, students can learn the connections between the arithmetic concept of ratio and the algebraic idea of linear slope. A related activity involving similarity of rectangles was created by Boester and Lehrer (2008). They gave students rectangles to sort into groups by similarity. The students then placed the similar rectangles onto the coordinate plane and saw that the corresponding vertices of rectangles similar to each other formed lines. The instructors then asked students to record the width and height data as an original rectangle was enlarged to a similar rectangle, prompting students to pay attention to the ratio of the "up" and the "over", thus revealing a growing linear relationship. These activities are examples of what might be contained in the curricula which this study envisions.

The results of this study contribute to a development of proportional reasoning that encompasses finding a qualitative measure of steepness. Although middle grades students encounter physical inclines on a regular basis, finding productive ways to measure these inclines' steepness is still challenging. Since students' levels of proportional reasoning and their levels of steepness attainment are related, future research could focus on developing curricula presenting proportions, steepness of inclines in a variety of settings, and slopes of lines concurrently, as well as examining the kinds of teacher support needed to help students learn these concepts in a connected way.

References

Adjiage, R., & Pluvinage, F. (2007). An experiment in teaching ratio and proportion. *Educational Studies in Mathematics, 65*, 149-175.

Behr, M., Wachsmuth, I., Post, T., & Lesh, R. (1984). Order and equivalence of rational numbers: A clinical teaching experiment. *Journal for Research in Mathematics Education, 15*(5), 323-341.

Boester, T. & Lehrer, R. (2008). Visualizing algebraic reasoning. In Kaput, J, Carraher, D. & Blanton, M. (Eds), *Algebra in the Early Grades (pp. 211-234).*New York, NY: Taylor & Francis Group.

Boyer, T., Levine, S., & Huttenlocher, J. (2008). Development of proportional reasoning: Where young children go wrong. *Developmental Psychology, 44*(5), 1478-1490.

Bright, G., Joyner, J., & Wallis, C. (2003). Assessing proportional thinking. *Mathematics Teaching in the Middle School, 9*(3), 166-172.

Brown, M., Hart, K., Kerslake, D., Kuchemann, D., Johnson, D., Ruddock, G., et al. (1981). *Secondary school children's understanding of mathematics: A report of the mathematics components of the Concepts in Secondary Science and Mathematics Programme*. London: Social Science Research Council.

Cheng, D., & Sabinin, P. (2008). *Elementary students' conceptions of steepness.* Paper presented at the Joint Meeting of the 32nd Conference of the International Group for the Psychology of Mathematics Education, and the XXX North American Chapter, Morelia, Michoacán, México.

Cheng, D., & Sabinin, P. (2009). *Transition from additive to proportional reasoning in preparation for learning about slope.* Paper presented at the American Educational Research Association Annual Meeting, San Diego, CA.

Cheng, I. (2010). Fractions: A new slant on slope. *Mathematics Teaching in the Middle School, 16(1), 34-41.*

De Bock, D., Verschaffel, L., & Janssens, D. (1998): The predominance of the linear model in secondary school students' solutions of word problems involving length and area of similar plane figures. Educational Studies in Mathematics, *35*, 65−85.

Gonzalez, P., Williams, T., Jocelyn, L., Roey, S., Kastberg, D., & Brenwald, S. (2008). *Highlights from TIMSS 2007: Mathematics and Science Achievement of U.S. Fourth-and Eighth-Grade Students in an International Context*. National Center for Education Sstatistics, Institute of Education Sciences, U.S. Department of Education. Washington, DC.

Hattikudur, S., Prather, R., Asquith, P., Knuth, E., & Nathan, M. (2012). Constructing graphical representations: Middle schoolers' intuitions and developing knowledge about slope and y-intercept. *School science and mathematics, 112*, 230-240.

Halford, G. S., Andrews, G., Dalton, C., Boag, C., & Zielinski, T. (2002). Young children's performance on the Balance Scale: The influence of relational complexity. *Journal of Experimental Child Psychology, 81*, 417–445.

Harel, G., Behr, M., Post, T., & Lesh, R. (1991). Variables affecting proportionality: Understanding of principles, formation of quantitative relations, and multiplicative invariance. In F. Furinghetti (Ed.) *Proceedings of PME XV Conference* (pp. 125-133). Assisi, Italy: PME.

Harel, G., Behr, M., Lesh, R., & Post, T. (1994). Invariance of ratio: The case of children's anticipatory scheme for constancy of taste. *Journal for Research in Mathematics Education, 25*(4), 324-345.

Harel, G., Behr, M., Post, T., & Lesh, R. (1994). The impact of the number type on the solution of multiplication and division problems: Further considerations. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics*, (pp. 365-388). Albany, NY: State University of New York Press.

Hart, K. (1981). Ratio and proportion. In K. Hart (Ed.), *Children's understanding of mathematics: 11-16* (pp. 88-101). East Sussex, England: Antony Rowe Publishing Services.

Inhelder, B., & Piaget, J. (1958). *The growth of logical thinking from childhood to adolescence; an essay on the construction of formal operational structures* (A. Parsons & S. Milgram, Trans.). New York: Basic Books.

Lamon, S. (1993). Ratio and proportion: Connecting context and children's thinking. *Journal for Research in Mathematics Education*, *24*(1), 41-61.

Lobato, J. (1996). *Transfer reconceived: How "sameness" is produced in mathematical activity*. University of California, Berkeley, Berkeley, CA.

Lobato, J., Ellis, A. B., & Munos, R. (2003). How ‘focusing phenomena’ in the instructional environment support individual students’ generalizations. *Mathematical Thinking and Learning, 5*, 1-36.

Lobato, J. & Ellis, A. (2010). Developing essential understnading of ratios, proportions & proportional reasoning: Grades 6-8. Reston, VA: National Council of Teachers of Mathematics.

Lobato, J., & Siebert, D. (2002). Quantitative reasoning in a reconceived view of transfer. *Journal of Mathematical Behavior, 21*, 87-116.

Lobato, J., & Thanheiser, E. (2002). Developing understanding of ratio-as-measure as a foundation for slope. In B. Litwiller & G. Bright (Eds.), *Making sense of fractions, ratios, and proportions: 2002 yearbook* (pp. 162-175). Reston, VA: National Council of Teachers of Mathematics.

Marshall, S. (1993). Assessment of rational number understanding: A schema-based approach. In T. Carpenter, E. Fennema & T. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 261-288). Hillsdale, NJ: Lawrence Erlbaum Associates.

Mitchelmore, M. C., & White, P. (2000). Development of angle concepts by progressive abstraction and generalisation. *Educational Studies in Mathematics, 41*(3), 209-238.

Moschkovich, J. N. (1996). Moving up and getting steeper: Negotiating shared descriptions of linear graphs. *Journal of the Learning Sciences*, *5*(3), 239-277.

Moschkovich, J. N. (1998). Resources for refining mathematical conceptions: Case studies in learning about linear functions. *Journal of the Learning Sciences, 7*(2), 209-237.

Moss, J., & Case, R. (1999). Developing children's understanding of rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education, 30*(2), pp. 122-147.

Moyer, J. C., Cai, J., & Grampp, J. (1997). The gift of diversity in learning through mathematical exploration. In J. Trentacosta (Ed.), *Multicultural and gender equity in the mathematics classroom: 1997 Yearbook of the National Council of Teachers of Mathematics* (pp. 151-163). Reston, VA: NCTM.

Noelting, G. (1980a). The development of proportional reasoning and the ratio concept part II: Problem-structure at successive stages; Problem-solving strategies and the mechanism of adaptive restructuring. *Educational Studies in Mathematics, 11*(3), 331-363.

Noelting, G. (1980b). The development of proportional reasoning and the ratio concept part 1: Differentiation of the stages. *Educational Studies in Mathematics, 11*(2), 217-253.

Schoen, H., & Hirsch, C. (2003). The Core Plus Mathematics Project: Perspectives & student achievement. In S. Senk & D. Thompson (Eds.), *Standards-oriented school mathematics curricula: What are they? What do students learn?* (pp. 311-344). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Siegler, R. (1976). Three aspects of cognitive development. *Cognitive psychology, 8*, 481-520.

Simon, M. A., & Blume, G. W. (1994). Mathematical modeling as a component of understanding ratio-as-measure: A study of prospective elementary teachers. *Journal of Mathematical Behavior, 13*(2), 183-197.

Stanton, M. & Moore-Russo, D. (2012). Conceptualizations of slope: A review of state standards. *School Science and Mathematics, 112,* 270-277.

Streefland, L. (1991). *Fractions in realistic mathematics education: A paradigm of developmental research.* Dordrecht, The Netherlands: Kluwer Academic Publications.

Stump, S. (1999). Secondary mathematics teachers' knowledge of slope. *Mathematics Education Research Journal, 11*(2), 124-144.

Stump, S. (2001). High school precalculus students' understanding of slope as measure. *School Science and Mathematics, 101* (2), 81-89.

Teuscher, D. & Reys, R. (2010). Slope, rate of change, and steepness: Do students understand these concepts? *Mathematics Teacher, 103*(7), 519-524.

Tourniaire, F., & Pulos, S. (1985). Proportional reasoning: A review of the literature. *Educational Studies in Mathematics, 16*(2), 181-204.

Yerushalmy, M. (1997). Designing representations: Reasoning about functions of two variables. *Journal for Research in Mathematics Education, 28*(4), 431-466.

Acknowledgments

Many thanks to Dr. Tim Heeren of Boston University and Dr. Polina Sabinin of Bridgewater State College for help on the statistical portion of this study.

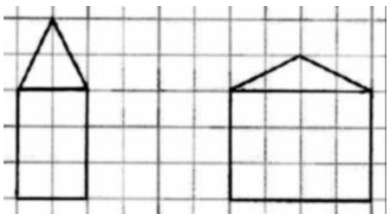
Appendix A
Steepness Test

This is a survey about steepness.
What does it mean to say that something is very steep?

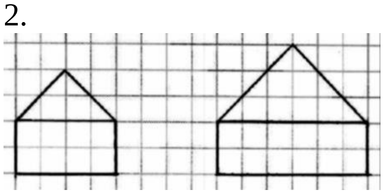
In the next problems you will decide on which roof, set of stairs, or line is steeper. If both have the same steepness, circle them both.

Circle the roof that is steeper. If both have the same steepness, circle them both.

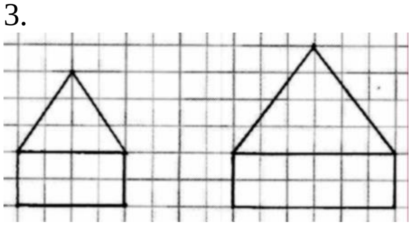
1.



2.

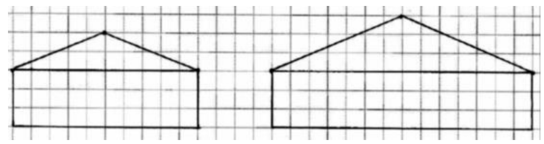


3.

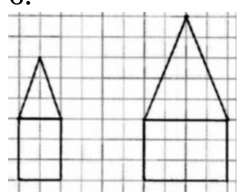


4.

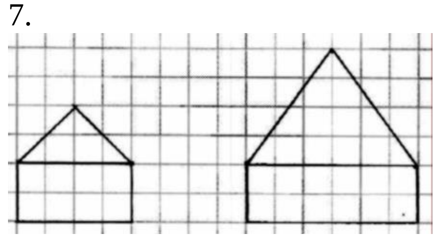
5.



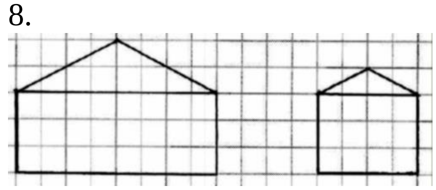
6.



7.

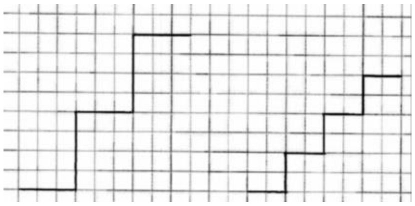


8.

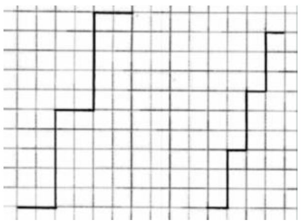


Circle the set of stairs that are steeper. If both have the same steepness, circle them both.

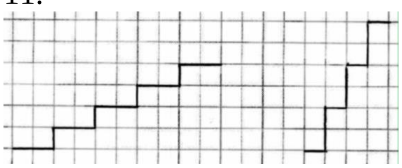
9.



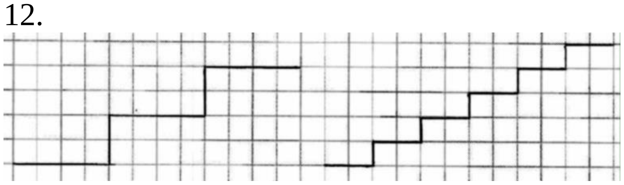
10.



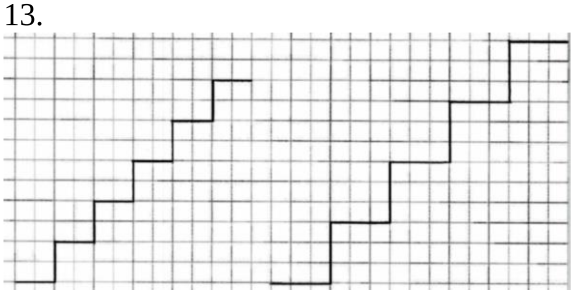
11.



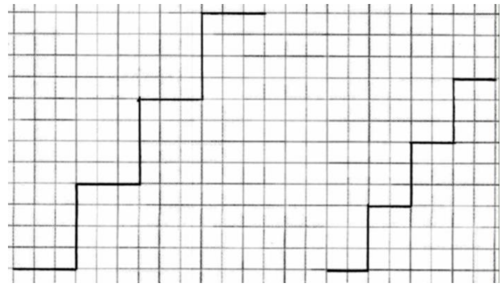
12.



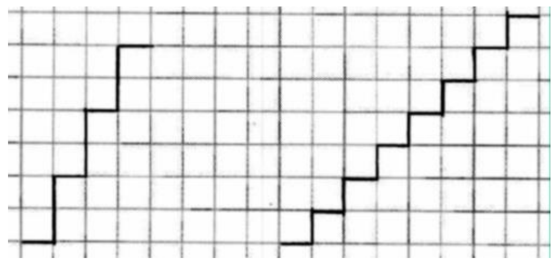
13.



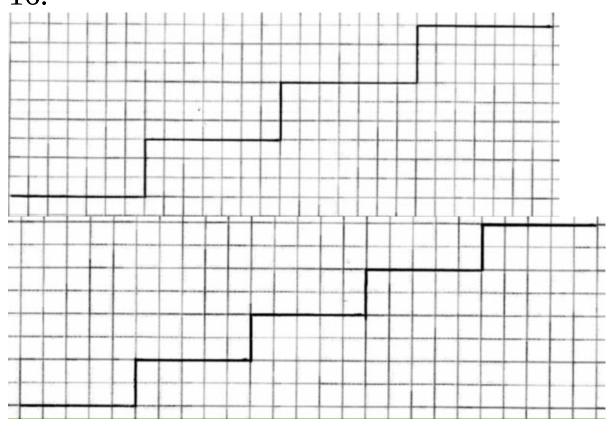
14.



15.

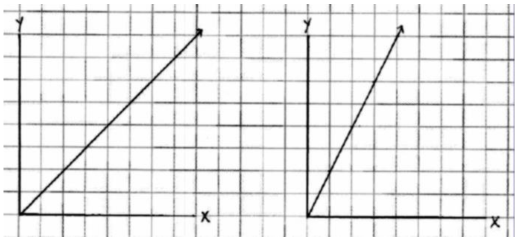


16.

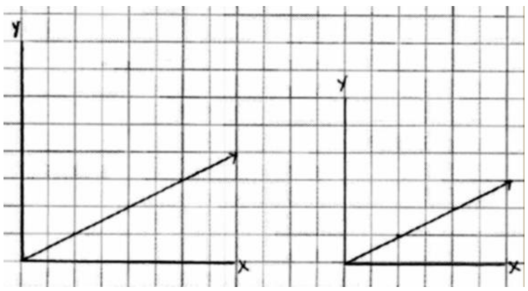


Circle the line that is steeper. If both have the same steepness, circle them both.

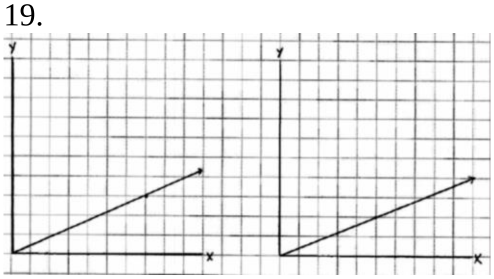
17.



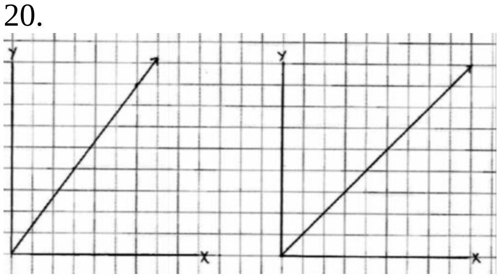
18.



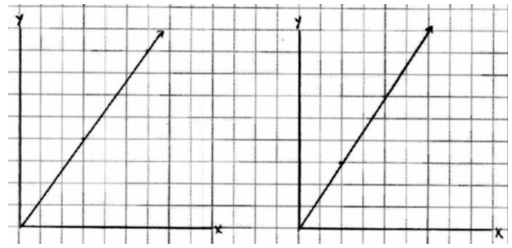
19.



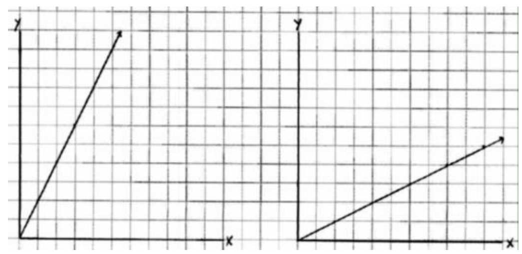
20.



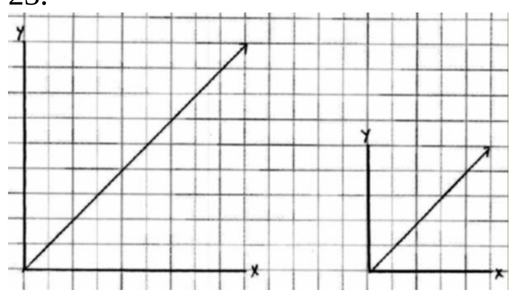
21.



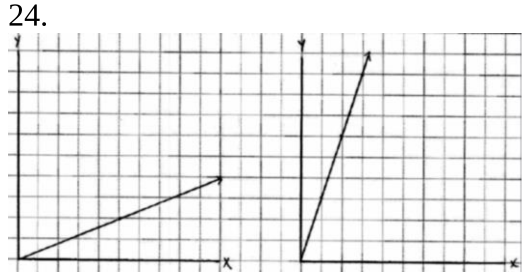
22.



23.



24.



THIS IS THE END OF THE SURVEY!
THANK YOU!